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Effects of anomalous magnetic moment in the quantum motion of neutral particle in magnetic and electric fields produced by a linear source in a conical spacetime

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Abstract: In this paper we analyse the effect of the anomalous magnetic moment on the non-relativistic quantum motion of a neutral particle in magnetic and electric fields produced by linear sources of constant current and charge density, respectively.

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Contents

1. Introduction

The role played by topology in the physical properties of a variety of system has been analysed in different areas as gravitation theory and condensed matter physics, for example. Topological defects appears in classical filed theory, as consequence of spontaneously broken of gauge symmetry as monopoles, strings and wall [1]. In condensed matter physics they are vortices in superconductors or superfluid and domain wall in magnetic materials [2], and dislocations or disclinations in disordered solid or liquid crystal [3]. Cosmic strings and disclinations are linear defects that changes the topology of the the medium in a similar way. In fact the similarity between these two objects goes beyond the topology: for some applications both kind of defects can be treated by the same geometrical methods [4].

Although the spacetime produced by a thin, infinite, straight cosmic string is locally flat [5], globally it is not. In fact the lack of global flatness is responsible for the existence of an induced electrostatic self-interaction on a point charged particle [6, 7], and also an induced electrostatic and magnetostatic self-interaction on a long straight wire which present constant linear density of charge and current parallel to the topological defect [8], gravitational analog of the Aharonov-Bohm effect [9], among others [10].

The metric tensor associated with infinite straight cosmic string lying along the $z-$axis can be given in cylindrical coordinate system by the following line element:

$$d^2s = -c^2dt^2 + dr^2 + \alpha^2r^2d\phi^2 + dz^2 \, , \tag{1.1}$$

where $r \geq 0$ and $0 \leq \phi \leq 2\pi$. The parameter α is given in terms of the linear mass density of the cosmic string μ by $\alpha = 1 - 4\mu/c^2$, which is smaller than unity. The linear defect in crystal liquid disclination, is obtained by either removing or inserting a wedge of material. Being σ the angle that defines the wedge, the metric of the disclinated medium is, in cylinder coordinates, similar to (1.1) according to the geometric approach given in Ref. [4], however $\alpha = 1 + \sigma/2\pi$. The space section corresponding to a conical one and the only nonzero components of the Riemann curvature tensor and Ricci tensor are give by:

$$R_{12}^{12} = R_1^1 = R_2^2 = 2\pi \frac{1-\alpha}{\alpha}\, \delta^2(\vec{r}) \,, \tag{1.2}$$

where $\delta^2(\vec{r})$ is the two dimensional delta function in flat space [11]. From the above expression follows that if $0 < \alpha < 1$, the defect presents a positive curvature and a planar deficit angle. On the other hand for disclinated medium the α parameter can be greater than unity for positive σ, and in this case the defect carries negative curvature and excess of planar angle, which corresponds to an anticonical spacetime.

As it was shown in [8] the presence of linear electric or magnetic source in the spacetime of a cosmic string parallel to the latter, become subject to a induced self-forces per unit length given by

$$\frac{\vec{F}_{\text{Ele.}}}{l} = \frac{(p-1)\lambda^2}{r}\hat{r} \tag{1.3}$$

and

$$\frac{\vec{F}_{\text{Mag.}}}{l} = -\frac{(p-1)}{r}\frac{j_0^2}{c^2}\hat{r} \,, \tag{1.4}$$

where $p = 1/\alpha$, λ the linear electric charge density and j_0 the current. We can see that for p bigger than unity the electric self-force becomes repulsive and the magnetic one attractive. For p smaller than unity the opposite situation occur.

The electric and magnetic fields produced by these two linear sources can be obtained as shown below:

(*a*) **Electric field.** Let us first calculate the scalar potential produced by the uniform linear charge density. In general the scalar potential is given by the Poison equation

$$\nabla^2 \Phi(\vec{r}) = -4\pi\rho(\vec{r}) \,, \tag{1.5}$$

where in this case $\rho(\vec{r}) = \lambda\delta^2(\vec{r} - \vec{r}_0)$. Calculating the solution to the Green function in the section $z = \ t = \ const.$

$$\nabla^2 G_p^2(\vec{r}, \vec{r}_0) = -4\pi\delta^2(\vec{r} - \vec{r}_0) \,, \tag{1.6}$$

one finds

$$G_p^2(\vec{r}, \vec{r}_0) = -\ln\left[r^{2p} + r_0^{2p} - 2(rr_0)^p \cos(\phi - \phi_0)\right] . \tag{1.7}$$

So the scalar potential reads

$$\Phi(\vec{r}) = \lambda\, G_p^2(\vec{r}, \vec{r}_0) \,. \tag{1.8}$$

The electric field is obtained taking its negative gradient: $\vec{E} = -\vec{\nabla}\Phi(\vec{r})$.

(*b*) **Magnetic field.** The vector potential associated with this linear current can be calculated by

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^2r\ G_p^2(\vec{r},\vec{r'})\ \vec{J}(\vec{r})\,, \tag{1.9}$$

with $\vec{J}(\vec{r}) = j_0\hat{z}\delta^2(\vec{r'}-\vec{r}_0)$. Substituting this expression inside the integral above we obtain:

$$\vec{A}(\vec{r}) = \frac{j_0}{c}\hat{z}\ G_p^2(\vec{r},\vec{r}_0)\,. \tag{1.10}$$

Finally the magnetic field can now be calculated by taking its rotational: $\vec{B} = \vec{\nabla}\times\vec{A}$.

The propose of this paper is to analyse the non-relativistic quantum motion of an uncharged spin-1/2 particle with an anomalous moment interacting with the electric and magnetic fields produced by a linear electric and magnetic sources in the cosmic string spacetime. In order to do that we shall consider the non-relativistic limit of the modified Dirac equation in this spacetime. Two different situations will be considered: (i) For $p > 1$, the electric current will be attracted to the cosmic string (disclination), while the linear charge distribution will be repelled by it. As the final configuration we expected that the current will be superposed to the cosmic string. Taking $r_0 = 0$ in (1.10) and its rotational, we obtain an azimuthal magnetic field

$$\vec{B}(\vec{r}) = \frac{2j_0\ p}{cr}\hat{\phi}\,. \tag{1.11}$$

(ii) For the case where $p < 1$, the linear charge distribution will be attracted to the disclination and the current repelled. In this case we expected as final configuration to have the linear charge superposed to the linear defect. The electric field produced by this configuration can also be obtained by taking the negative gradient of the scalar potential considering $r_0 = 0$. This field is radial as shown below:

$$\vec{E} = \frac{2\lambda\ p}{r}\hat{r}\,. \tag{1.12}$$

The effect of an arbitrary electromagnetic field in the quantum motion of a neutral particle with anomalous magnetic and electric moments has been analysed by Anandan [12]. There, using the non-relativistic limit of the Dirac equation Anandan was able to shown that the two-component wave-function associated with the particle acquires an Aharonov-Bohm phase. The induced gauge field presents a non-abelian character which makes the analysis of the quantum and classical motion, and the phase shift richer. More recently Audretsch and Skarzhinsky [13] have analysed the Aharonov-Bohm scattering of a charged particles and neutral atoms with induced electric electric dipole moments in absorbing medium using the Schrödinger equation in the $z = 0$ plane. The analysis of a charged particle which presents anomalous magnetic moment in the presence of three different configurations of magnetic fields along the $z-$direction, has been developed in the context of the non-relativistic Schrödinger equation [14], and also relativistic Dirac equation [15] by Bordag and Voropaev. There they shown that bound states are present if the magnetic moment is anti-parallel to the magnetic fields.

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This paper is organized as follows: in section 2 we develop the formalism needed to analyse the problem under consideration, i.e., the non-relativistic quantum motion of uncharged particle with anomalous magnetic moment in the presence of magnetic and electric fields in a conical spacetime. Some mathematical details will be introduced in order to derive the non-relativistic limit of the modified Dirac equation as a Schrödinger-Pauli one, to analyse two different physical systems. We shall see that the equation is a generalization of the similar one obtained by Anandan in a conical spacetime. In section 3 we explicitly analyse the quantum motion of an uncharged two-component spinor field in azimuthal and radial, magnetic and electric fields, respectively. Because of the presence of the anomalous moment, we obtain for the first case, two coupled second order differential equations. Although looking simple this set of differential equations, we were not able obtain an exact solution. However, for large total angular quantum number, approximated solutions can be provided. Moreover, for bound states solutions we could obtain the respective quantized energy. For the second case, considering only electric field, we do not find bound states, only scattering ones take place and a phase shift can be provided. As we shall see, for this case the scattering amplitude associated with both components of the spinor field depend on the parameter α, which codifies the conical structure of the spacetime, and also on the anomalous coupling factor κ. We leave for section 4 our most important remarks about our analysis of the system. In the appendix we present some specific calculations used to obtain the non-relativistic limit of the modified Dirac equation in a conical spacetime in a more familiar and workable way.

2. The equation of motion

The electromagnetic interactions of mesons and nucleons depend strongly on their anomalous electric and magnetic moments. For specific case of the neutron its magnetic moment is $-1.91\mu_B$, whereas the free Dirac equation predicts a zero value. It is possible to include the effect of such interaction in the analysis of quantum motion if we abandon the principle of minimal coupling [16]. For a flat spacetime instead of consider only the minimal electromagnetic coupling, it is added an extra dipole interaction term:

$$\not{\pi} = \gamma^{(a)}\left(i\hbar\partial_{(a)} - \left(\frac{e}{c}\right)A_{(a)}\right) \quad \longrightarrow \quad \not{\pi} - \kappa\frac{\mu_B}{2\hbar c}\sigma^{(a)(b)}F_{(a)(b)}\,, \tag{2.1}$$

where κ is the anomalous factor, $\sigma^{(a)(b)} = 1/2[\gamma^{(a)},\gamma^{(b)}]$, $\mu_B = e\hbar/2mc$ the Bohr magneton and $\gamma^{(a)}$ the Dirac matrices.

However for a non-flat spacetime, we have to include the spin connection in the differential operator and define the respective Dirac matrices in this manifold. Taking a basis tetrad $e^{\mu}_{(a)}$ satisfying the condition

$$e^{\mu}_{(a)}e^{\nu}_{(b)}\eta^{(a)(b)} = g^{\mu\nu}\,, \tag{2.2}$$

the spin connection Γ_μ can be expressed by

$$\Gamma_\mu = -\frac{1}{4}\gamma^{(a)}\gamma^{(b)}e^{\nu}_{(a)}e_{(b)\nu;\mu}\,, \tag{2.3}$$

and the γ matrices in the curved spacetime by $\gamma^{\mu} = e^{\mu}_{(a)}(x)\gamma^{(a)}$. In this way the modified Dirac equation in the curved space reads:

$$\left[i\hbar\gamma^{\mu}(\partial_{\mu} + \Gamma_{\mu}) - \frac{e}{c}\gamma^{\mu}A_{\mu} - \kappa\frac{\mu_B}{2c}\sigma^{\mu\nu}F_{\mu\nu} - mc\right]\Psi(x) = 0\,. \tag{2.4}$$

Because we have the intention to analyse the non-relativistic limit of this Dirac equation *a la* procedure presented in [16], let us first to express it in terms of an hamiltonian formalism. After some intermediate steps we get:

$$\hat{H}\Psi(x) = i\hbar\partial_t\Psi(x)\,, \tag{2.5}$$

with

$$\hat{H} = c\alpha^i(-i\hbar\partial_i + \frac{e}{c}A_i) + eA_0 - ic\hbar\gamma^0\gamma^{\mu}\Gamma_{\mu} + \kappa\frac{\mu_B}{2}\gamma^0\sigma^{\mu\nu}F_{\mu\nu} + \beta mc^2\,, \tag{2.6}$$

where we have assumed that the time-time component of the metric tensor is equal to -1, and defined $\alpha^i = \gamma^0\gamma^i$ and $\beta = \gamma^0$ as usual. The $\gamma-$matrices obey the anticommutator relations $\{\gamma^{\mu},\ \gamma^{\nu}\} = -2g^{\mu\nu}$.

Now after this brief introduction let us specialize in the cosmic string spacetime whose the metric tensor components are given by (1.1). Adopting the following representation to the $\gamma-$matrices in a flat spacetime

$$\gamma^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^{(a)} = \begin{pmatrix} 0 & \sigma^{(a)} \\ -\sigma^{(a)} & 0 \end{pmatrix}, \tag{2.7}$$

with $\sigma^{(a)}$ being the standard Pauli matrices and a particular basis tetrad

$$e^{\mu}_{(a)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi/\alpha r & 0 \\ 0 & \sin\phi & \cos\phi/\alpha r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.8}$$

For the cosmic string spacetime

$$\gamma^{\mu}\Gamma_{\mu} = -\frac{(\alpha^{-1} - 1)}{2r}\gamma^r\,, \tag{2.9}$$

with

$$\gamma^r = \cos\phi\gamma^{(1)} + \sin\phi\gamma^{(2)} = \begin{pmatrix} 0 & \sigma^r \\ -\sigma^r & 0 \end{pmatrix}. \tag{2.10}$$

Moreover the $\alpha-$matrices are now written as

$$\alpha^i = e^i_{(a)}\begin{pmatrix} 0 & \sigma^{(a)} \\ \sigma^{(a)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \tag{2.11}$$

where $\sigma^i = (\sigma^r, \sigma^{\phi}, \sigma^z)$ are the Pauli matrices in cylindrical coordinates obtained from the basis tetrad (2.8).

Now let us study the non-relativistic limit of the modified Dirac equation. In this limit we have to extract from the wave-function its temporal behavior consequence of the rest energy. So writing

$$\Psi = e^{-i(mc^2/\hbar t)}\begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \tag{2.12}$$

we obtain the following differential equation:

$$c\sigma^i\pi^i\begin{pmatrix}\chi\\ \varphi\end{pmatrix}+i\hbar\frac{(\alpha^{-1}-1)}{2r}c\sigma^r\begin{pmatrix}\chi\\ \varphi\end{pmatrix}+eA_0\begin{pmatrix}\varphi\\ \chi\end{pmatrix}+i\kappa\mu_B\frac{d\Phi(r)}{dr}\sigma^r\times$$
$$\times\begin{pmatrix}\chi\\ -\varphi\end{pmatrix}-\kappa\mu_B F_{13}\alpha r\sigma^\phi\begin{pmatrix}\varphi\\ -\chi\end{pmatrix}-2mc^2\begin{pmatrix}0\\ \chi\end{pmatrix}=i\hbar\partial_t\begin{pmatrix}\varphi\\ \chi\end{pmatrix}, \tag{2.13}$$

where $\pi^i = -i\hbar\partial_i + (e/c)A_i$. The "small" component of the wave function, χ, can be related with its "large" one, φ, by

$$\chi=\frac{1}{2mc}\sigma^i\left[-i\hbar(\partial_i+\Gamma_i)+\left(\frac{e}{c}\right)A_i+i\kappa\left(\frac{\mu_B}{c}\right)E_i\right]\varphi\,. \tag{2.14}$$

It is worth to mention that the approach developed here allows to consider the two different physical configurations at the same time. For the case where the magnetic interaction will be considered we discard the electric field in our equations, and the opposite procedure if we are considering electric interaction. Moreover, because vector potential posses only component along the $z-$direction and the electric field is radial, and because $g_{11} = g_{33} = 1$, we have $A^z = A_z$ and $E^r = E_r$, so we can see that $\sigma^r E^r = \sigma^i E_i$. We are writing the general formalism above in order to be complete, however, later we shall consider uncharged particle only, and these observations will not be essential anymore.

The remaining equation is:

$$c\sigma^i\left[-i\hbar(\partial_i+\Gamma_i)+\left(\frac{e}{c}\right)A_i-i\kappa\left(\frac{\mu_B}{c}\right)E_i\right]\chi+eA_0\varphi-\mu_B\kappa F_{13}\alpha r\sigma^\phi\varphi=i\hbar\partial_t\varphi\,. \tag{2.15}$$

Now substituting (2.14) into the above equation we obtain:

$$\hat{H}_{\rm NR}\varphi=i\hbar\partial_t\varphi\,, \tag{2.16}$$

where

$$\hat{H}_{\rm NR}=\frac{\sigma^i(\Pi_i-i\kappa(\mu_B/c)E_i)\sigma^j(\Pi_j+i\kappa(\mu_B/c)E_j)}{2m}+eA_0-\mu_B\kappa\alpha rF_{13}\sigma^\phi\,, \tag{2.17}$$

where now $\Pi_i = -i\hbar(\partial_i + \Gamma_i) + (e/c)A_i$. In the appendix we present specific calculations used to write the above hamiltonian in a more familiar and workable way. Our final expression to it is:

$$\hat{H}_{\rm NR}=\frac{1}{2m}\frac{1}{\sqrt{g^{(3)}}}\left(\Pi_i-\kappa\frac{\mu_B}{c}(\vec{E}\times\vec{\sigma})_i\right)\left[g^{ij}\sqrt{g^{(3)}}\left(\Pi_j-\kappa\frac{\mu_B}{c}(\vec{E}\times\vec{\sigma})_j\right)\right]+$$
$$+\frac{\kappa\mu_B\hbar}{2mc}(\vec{\nabla}\cdot\vec{E})+\frac{\hbar e}{2mc}\alpha r\vec{B}\cdot\vec{\sigma}-eA^0+\mu_B\kappa\alpha r\vec{B}\cdot\vec{\sigma}-\frac{\kappa^2\mu_B^2}{2mc^2}\vec{E}^2\,, \tag{2.18}$$

where $g^{(3)}$ is the determinant of the spatial component of (1.1).

Having found the general expression to the non-relativistic hamiltonian which governs the motion of a charged particle with anomalous moment in the conical spacetime, and in presence of a cylindrically symmetric radial and azimuthal electric and magnetic fields, respectively, let us specialize in two distinct situations already mentioned. So the first steps is to set the electric charge equal to zero. The second one in fact involves two different approaches which will be considered independently in the next section.

3. The non-relativistic motion

3.1 Motion in a azimuthal magnetic field

This situation happens when the linear uniform electric current is running along the $z-$axis in the cosmic string spacetime. For this case (2.18) reads:

$$\hat{H}_{\rm NR} = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{g^{(3)}}}\nabla_i\left(g^{ij}\sqrt{g^{(3)}}\nabla_j\right) + \mu_B\kappa\vec{B}\cdot\vec{\sigma}\,, \tag{3.1}$$

where $\nabla_i = \partial_i + \Gamma_i$. For the specific basis tetrad used here, the only non-vanishing spin connection is:

$$\Gamma_\varphi = i\frac{(1-\alpha)}{2}\sigma^z\,. \tag{3.2}$$

As it has been pointed out by Anandan in [12], here we also can interpret the above hamiltonian as the operator which governs the motion of a "charged" particle coupled with SU(2) gauge potential $A_\mu = A_\mu^a\sigma^a$ in the conical spacetime. Comparing the above expressions with the ordinary definition to the operator $\pi^i = -i\hbar\partial_i + q/cA_i$, and the electrostatic interaction $-qA_0$, we can identify the "gauge potential" as:

$$A_0^a = \mu_B\left(\frac{\kappa}{q}\right)B^a \tag{3.3}$$

and

$$A_i^a = \frac{\hbar c(1-\alpha)}{2q}\delta_i^\phi\,\delta_3^a\,, \tag{3.4}$$

where the latter has purely geometric character. The two-component iso-spinor reads:

$$\phi = \begin{pmatrix}\phi_+\\ \phi_-\end{pmatrix}. \tag{3.5}$$

After this identification, we pass now to develop the above hamiltonian, which can be written as:

$$\hat{H}_{\rm NR} = -\frac{\hbar^2}{2m}\nabla^2_{(\alpha)} - \frac{i\hbar}{2m}\frac{(1-\alpha)}{\alpha^2r^2}\sigma^z\partial_\phi + \frac{\hbar^2}{2m}\frac{(1-\alpha)^2}{4\alpha^2r^2} - \frac{2\mu_B\kappa j_0}{c}\sigma^\phi\,, \tag{3.6}$$

where $\nabla^2_{(\alpha)} = \partial_r^2 + 1/r\partial_r + (1/\alpha^2r^2)\partial_\phi^2 + \partial_z^2$, is the Laplace-Beltrami operator in the conical space. In the above expression we have substituted the explicit expression to the magnetic field: eq. (1.11). Because this system presents a invariance under boosts along the $z-$direction the stationary states can be written as $\Psi(t,\vec{x}) = e^{-iEt/\hbar}e^{ik_zz}u(r,\theta)$. Substituting this wave-function into the Schrödinger-Pauli equation we get the self-energy equation:

$$\hat{H}_{\rm Eff}u(r,\theta) = Eu(r,\theta)\,, \tag{3.7}$$

with

$$\begin{aligned}\hat{H}_{\rm Eff} = &-\frac{\hbar^2}{2m}\left[\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{\alpha^2r^2}\partial_\phi^2\right] + \frac{\hbar^2k_z^2}{2m} + \frac{i\hbar^2}{2m}\frac{(1-\alpha)}{\alpha^2r^2}\sigma^z\partial_\phi + \\ &+\frac{\hbar^2}{2m}\frac{(1-\alpha)^2}{4\alpha^2r^2} - \frac{2\mu_B\kappa j_0}{c}\sigma^\phi\,.\end{aligned} \tag{3.8}$$

For this system a total angular momentum operator can be defined as the sum of the orbital and iso-spin parts

$$\hat{J} = -i\partial_\theta + \left(\frac{1}{2}\right)\sigma^z \,. \tag{3.9}$$

It can be show that $\hat{J}$ commutes with the effective hamiltonian. So it is possible to express the eigenfunctions of the effective two dimensional hamiltonian, $\hat{H}_{\text{Eff}}$, in terms of the eigenfunction of $\hat{J}$. These functions have the form

$$u_j(r,\theta) = \frac{e^{ij\theta}}{\sqrt{r}}\begin{pmatrix} u_+(r)\; e^{-i\theta/2} \\ iu_-(r)\; e^{i\theta/2}\end{pmatrix}, \tag{3.10}$$

with $j = n + 1/2 = \ \pm 1/2, \pm 3/2, \ldots$.

Substituting (3.8) and (3.10) into (3.7) we obtain, after a few steps, a set of two coupled radial differential equations:

$$u''_+(r) - \frac{j(j-\alpha)}{\alpha^2 r^2}u_+(r) - \frac{4m\mu_B\kappa j_0}{\alpha\hbar^2 cr}u_-(r) + k^2 u_+(r) = 0 \tag{3.11}$$

and

$$u''_-(r) - \frac{j(j+\alpha)}{\alpha^2 r^2}u_-(r) - \frac{4m\mu_B\kappa j_0}{\alpha\hbar^2 cr}u_+(r) + k^2 u_-(r) = 0\,, \tag{3.12}$$

where $k^2 = 2m\mathcal{E}/\hbar^2$, being $\mathcal{E} = E - \hbar^2 k^2/2m$. Setting $\kappa = 0$ we obtain two non-coupled differential equations. Moreover, this system is symmetric under the transformation $j \to -j$ and $u_+ \rightleftharpoons \ u_-$. The asymptotic behavior for both solutions can be obtained by taking $r \to \ \infty$. For both equations we get

$$u''_\pm + k^2 u_\pm = 0\,, \tag{3.13}$$

which gives rise to possibilities: (i) scattering states, $u_\pm \simeq e^{\pm ikr}$ and (ii) bound states if $k^2 = -k_B^2$ with $u_\pm \simeq \ e^{-k_B r}$. It is of our interest in this sub-section to analyse the possibility of this system to present bound states solutions. In this way, defining a dimensionless variable $z = k_B r$, we can rewrite the two differential equations above as:

$$\frac{d^2u_+}{dz^2} - \frac{j(j-\alpha)}{\alpha^2 z^2}u_+ - \frac{\delta}{z}u_- - u_+ = 0 \tag{3.14}$$

and

$$\frac{d^2u_-}{dz^2} - \frac{j(j+\alpha)}{\alpha^2 z^2}u_- - \frac{\delta}{z}u_+ - u_- = 0\,, \tag{3.15}$$

with

$$\delta = \frac{4m\mu_B\kappa j_0}{\alpha\hbar^2 k_B c}\,. \tag{3.16}$$

Defining

$$\Psi(r) = \begin{pmatrix} u_+(r) \\ u_-(r)\end{pmatrix}, \tag{3.17}$$

we can write these two equations above in terms of a 2×2 hermitian hamiltonian operator:

$$\hat{H}\Psi(r) = -\Psi(r)\,, \tag{3.18}$$

with

$$\hat{H} = -I_{(2)}\left(\frac{d^2}{dz^2} - \frac{j^2}{\alpha^2 z^2}\right) - \sigma^{(3)}\frac{j}{\alpha z^2} + \sigma^{(1)}\frac{\delta}{z}, \tag{3.19}$$

being $I_{(2)}$ the identity matrix. Notice that the above system resemble with the Schrödinger equation for a particle with mass $M = 1/2$ and $E = -1$ in the presence of a non-abelian Coulomb potential. This potential has strength δ and couples both components of the effective radial wave-function (3.17). Considering this analogy, the Coulomb potential will be attractive for negative value of δ and repulsive in the opposite case. Because we are interested to analyse the possibility of this system to present bound states, let us assume κ negative. This is what happens with the neutron, which presents $\kappa = -1.91$. Taking $\delta = -|\delta|$ in previous equations we get:

$$\frac{d^2u_+}{dz^2} - \frac{j(j-\alpha)}{\alpha^2 z^2}u_+ + \frac{|\delta|}{z}u_- - u_+ = 0 \tag{3.20}$$

and

$$\frac{d^2u_-}{dz^2} - \frac{j(j+\alpha)}{\alpha^2 z^2}u_- + \frac{|\delta|}{z}u_+ - u_- = 0\,. \tag{3.21}$$

Although looks simple this set of coupled differential equation, we could not obtain an exact solution for it. However our main interest in this paper is to investigate the possibility of this system to present bound states. In this sense we can proceed on this analysis to the case where the total angular quantum number $|j|$ is much grater than α. Accepting this situation, two distinct classes of solutions can be analyzed: $i)$ $u_+ = -u_-$ and $ii)$ $u_+ = u_-$. For the first class the effective Coulomb potential becomes repulsive in both differential equations, so no bound state can be formed. However for the second class, it becomes attractive making possible the existence of bound states. For the latter both equations can be written as:

$$\frac{d^2f(z)}{dz^2} - \frac{j^2}{\alpha^2 z^2}f(z) + \frac{|\delta|}{z}f(z) - f(z) = 0\,. \tag{3.22}$$

The solution to this equation regular at origin is:

$$f(z) = M_{|\delta|/2,\sqrt{1/4+j^2/\alpha^2}}(2z)\,, \tag{3.23}$$

where $M_{\lambda,\mu}(z)$ is the Whittaker function [17], which is expressed in terms of confluent hypergeometric function as:

$$M_{\lambda,\mu}(z) = z^{\mu+1/2}e^{-z/2}\Phi\left(\mu - \lambda + \frac{1}{2}, 2\mu+1, z\right). \tag{3.24}$$

In order to obtain bound states solution, one has to impose specific condition on the parameter $\mu - \lambda + 1/2$ to terminate the series, transforming it in a polynomial. This condition is to make this parameter equal to a non positive integer number. For our case this condition provides

$$\sqrt{\frac{1}{4} + \frac{j^2}{\alpha^2}} + \frac{1}{2} - \frac{|\delta|}{2} \approx \frac{|j|}{\alpha} + \frac{1}{2} - \frac{|\delta|}{2} = -n\,, \tag{3.25}$$

n being $0, 1, 2, \ldots$. Substituting the expression to δ given (3.16), $k_B = \sqrt{2m|\mathcal{E}|}/\hbar$ and after some intermediate steps we find:

$$\mathcal{E} = E - \frac{\hbar^2 k_z^2}{2m} = -\frac{8m\mu_B^2\kappa^2 j_0^2}{\alpha^2\hbar^2c^2(2n+1+2|j|/\alpha)^2} \,. \tag{3.26}$$

3.2 Motion in the radial electric field

This situation happens in disclinated medium with α bigger than unity. In this case a linear electric charge distribution may be localized along the $z-$axis. For this case (2.18) becomes:

$$\hat{H}_{\rm NR} = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{g^{(3)}}}\left(\nabla_i - i\frac{A_i}{\hbar}\right)\left[g^{ij}\sqrt{g^{(3)}}\left(\nabla_j - i\frac{A_j}{\hbar}\right)\right] - \frac{\kappa^2\mu_B^2}{2mc^2}\vec{E}^2 \,, \tag{3.27}$$

which can be seen as a hamiltonian that governs the motion of a SU(2) "charged" iso-spinor particle interacting with a non-abelian vector potential

$$A_i = c\hbar\frac{(1-\alpha)}{2q}\sigma^z\delta_i^\phi - \frac{a_i c}{q} \tag{3.28}$$

with

$$a_1 = 0\,, \qquad a_2 = -\frac{\kappa\mu_B\alpha r}{c}E^r\sigma^z\,, \qquad a_3 = \frac{\kappa\mu_B\alpha r}{c}E^r\sigma^\phi \tag{3.29}$$

and abelian scalar potential

$$A_0 = -\frac{\kappa^2\mu_B^2}{2mc^2q}\vec{E}^2 \tag{3.30}$$

in conical spacetime. q being the "electric" charge associated with these potentials. In (3.27) we have dropped the term $\vec{\nabla}\cdot\vec{E}$, which is proportional to the electric charge density of the source giving by a delta-function concentrated on the defect: $\vec{\nabla}\cdot\vec{E} = 2(\lambda/r)\delta(r)$. This procedure is allowed for wave-function that vanish at origin. However, as will be mentioned later, by constructing adjoint extension of the above hamiltonian, specific irregular solutions are accepted.

The complete expansion to the above hamiltonian contains many terms which can be conveniently grouped. Inside the brackets there appears a term exactly twice the similar one square in the electric field however with positive sign. So, after some intermediate steps the hamiltonian can be written as

$$\begin{aligned}\hat{H}_{\rm NR} = & -\frac{\hbar^2}{2m}\nabla^2_{(\alpha)} - \frac{i\hbar^2(1-\alpha)}{2m}\frac{\sigma^z}{\alpha^2r^2}\partial_\phi + \frac{\hbar^2}{2m}\frac{(1-\alpha)^2}{4\alpha^2r^2} - \frac{i\hbar\kappa\mu_B E^r\sigma^z}{mc\alpha r}\partial_\phi + \\ & + \frac{\hbar\kappa\mu_B E^r(1-\alpha)}{2mc\alpha r} + \frac{i\hbar\kappa\mu_B\alpha r E^r\sigma^\phi}{mc}\partial_z + \frac{\kappa^2\mu_B^2}{2mc^2}(E^r)^2 \,. \end{aligned} \tag{3.31}$$

As in the previous analysis this system presents a invariance under boosts along the $z-$direction, so its stationary states can be written as $\Psi(t,\vec{x}) = e^{-iEt/\hbar}e^{ik_zz}u(r,\theta)$. Substituting this wave-function into the Schrödinger-Pauli equation

$$\hat{H}_{\rm Eff}u(r,\theta) = Eu(r,\theta)\,, \tag{3.32}$$

we obtain

$$\hat{H}_{\rm Eff} = -\frac{\hbar^2}{2m}\left[\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{\alpha^2 r^2}\partial_\phi^2\right] - \frac{i\hbar^2(1-\alpha)}{2m}\frac{\sigma^z}{\alpha^2 r^2}\partial_\theta + \frac{\hbar^2}{2m}\frac{(1-\alpha)^2}{4\alpha^2 r^2} - \\ - \frac{i\hbar\kappa\mu_B E^r\sigma^z}{mc\alpha r}\partial_\phi + \frac{\hbar\kappa\mu_B E^r(1-\alpha)}{2mc\alpha r} - \frac{\hbar\kappa\mu_B\alpha r E^r\sigma^\phi k_z}{mc} + \frac{\kappa^2\mu_B^2}{2mc^2}(E^r)^2 . \qquad (3.33)$$

As in the previous analysis we can verify that the total angular operator $\hat{J}$ commutes with $\hat{H}_{\rm Eff}$ above, consequently we can use the same procedure expressing its eigenfunctions as exhibited in (3.10). Moreover, from the explicit form of the effective hamiltonian, we can observe that choosing $k_z = 0$, it becomes diagonal, consequently the upper and lower components of the wave-function do not interact among themselves. So, for simplicity only, let us consider this possibility. Finally taking into account all the observations mentioned, and after some intermediate steps we arrive at:

$$u''_+(r) - \frac{j(j-\alpha)}{\alpha^2 r^2}u_+(r) - \frac{\kappa\mu_B(2j-\alpha)}{\hbar c\alpha r}E^r u_+(r) - \frac{\kappa^2\mu_B^2(E^r)^2}{\hbar^2 c^2}u_+(r) = -k^2 u_+(r) \qquad (3.34)$$

and

$$u''_-(r) - \frac{j(j+\alpha)}{\alpha^2 r^2}u_-(r) + \frac{\kappa\mu_B(2j+\alpha)}{\hbar c\alpha r}E^r u_-(r) - \frac{\kappa^2\mu_B^2(E^r)^2}{\hbar^2 c^2}u_-(r) = -k^2 u_-(r) \qquad (3.35)$$

with $k^2 = 2mE/\hbar^2$. Substituting the expression to the electric field (1.12) and defining a new variable $z = kr$ we can notice that both equations take the form

$$\frac{d^2u_\pm(z)}{dz^2} - \frac{\gamma_\pm}{z^2}u_\pm(z) + u_\pm(z) = 0 , \qquad (3.36)$$

where

$$\gamma_\pm = \frac{j(j\mp\alpha)}{\alpha^2} \pm \frac{2\kappa\mu_B\lambda(2j\mp\alpha)}{\alpha^2\hbar c} + \frac{4\kappa^2\mu_B^2\lambda^2}{\alpha^2\hbar c^2} . \qquad (3.37)$$

The general solution for these equations are given in terms of Bessel functions: $u_\pm(z) = \sqrt{z}(A_{\nu_\pm}J_{\nu_\pm}(z) + B_{\nu_\pm}J_{-\nu_\pm}(z))$ with

$$\nu_\pm = \sqrt{\frac{1}{4} + \gamma_\pm} = \left|\frac{n}{\alpha} + \frac{1\mp\alpha}{2\alpha} \pm \frac{2\kappa\mu_B\lambda}{\alpha\hbar c}\right| , \qquad (3.38)$$

where we have substituted $j = n + 1/2$.

The scattering amplitude associated with this system can be evaluated by using partial-wave analysis. A few years ago, Hagen showed that, similarly with happens in the Aharonov-Bohm scattering for spin$-1/2$ particles by an idealized solenoid [18], the admissible solutions to the above equations are the regular one, unless $\nu_\pm$ be smaller than unity [19]. In this case, irregular solutions becomes the only ones acceptable. So, to calculate the scattering amplitude associated with this system by using partial-wave analysis, both kind of solutions must be taken into account. Defining $2\kappa\mu_B\lambda/(\hbar c) = N + \delta$, with N being an integer number and $0 \le \delta < 1$, the acceptable singular solutions are given by $n =$

$-N-1$ to $u_+(z)$ and by $n = N$ to $u_-(z)$. For both, the order of the singular Bessel function is

$$\nu = -\frac{1+\alpha}{2\alpha} + \frac{\delta}{\alpha} = -\beta\,, \tag{3.39}$$

provided $0 < \beta < 1$.[1]

The regular effective two dimensional wave function (3.10) is now written as:

$$u_j(r,\theta) = u_n(r,\theta) = e^{in\theta}\begin{pmatrix} c_+^n J_{\nu_+}(kr) \\ c_-^n J_{\nu_-}(kr)\; e^{i\theta} \end{pmatrix}, \tag{3.40}$$

where the coefficients c_n^+ and c_n^- can be determined by imposing that the complete wave-function given below, for large values of r, behaves as an incident plane wave coming from the right:

$$\psi(r,\theta) = {\sum}' u_n(r,\theta) + \begin{pmatrix} e^{-i(N+1)\theta} e^{i(\pi/2)\beta} J_{-\beta}(kr) \\ e^{i(N+1)\theta} e^{i(\pi/2)\beta} J_{-\beta}(kr) \end{pmatrix}. \tag{3.41}$$

In the above expression the prime on the summation indicates the exclusion of the irregular solutions. Using the asymptotic behavior for the Bessel functions, the coefficients c_n^+ and c_n^- are given by:

$$\begin{aligned} c_n^+ &= e^{-i(\pi/2)\nu_+} \\ c_n^- &= e^{-i(\pi/2)\nu_-} \end{aligned} \tag{3.42}$$

with $\theta = \pi + \theta'$.

In order to calculate the complete scattering amplitude associated with this system, it is more convenient to write (3.41) as shown below:

$$\psi(r,\theta) = \psi_R(r,\theta) + \psi_I(r,\theta)\,, \tag{3.43}$$

with

$$\psi_R(r,\theta) = \sum_{n=-\infty}^{\infty} u_n(r,\theta) \tag{3.44}$$

and

$$\psi_I(r,\theta) = \left(e^{i(\pi/2)\beta} J_{-\beta}(kr) - e^{-i(\pi/2)\beta} J_{\beta}(kr)\right)\begin{pmatrix} e^{-i(N+1)\theta} \\ e^{i(N+1)\theta} \end{pmatrix}. \tag{3.45}$$

So to obtain the complete scattering amplitude, two distinct contributions can be separately calculated.

The scattering amplitude associated with the regular solutions, $f_R(\theta)$, can be calculated by using the phases shift δ_n^+ and δ_n^-, obtained from the asymptotic behavior of each component of (3.44). It is:

$$f_R(\theta) = \frac{1}{\sqrt{2\pi i k}} \sum_{n=-\infty}^{\infty} e^{in\theta}\begin{pmatrix} e^{2i\delta_n^+} - 1 \\ e^{2i\delta_n^-} - 1 \end{pmatrix} = \begin{pmatrix} f^+(\theta) \\ f^-(\theta) \end{pmatrix}. \tag{3.46}$$

[1]The system of spin$-1/2$ particle with anomalous magnetic moment in presence of a idealized solenoid has been analyzed in [20] in a non-relativistic context. There by applying the method of von Neumann to provide a self-adjoint extension to the hamiltonian, they also shown that singular solutions to the wave-equation are accepted.

JHEP06(2004)016

with

$$
\begin{aligned}
\delta_n^+ &= -\frac{\pi}{2}\nu_+ + \frac{\pi}{2}|n| \\
\delta_n^- &= -\frac{\pi}{2}\nu_- + \frac{\pi}{2}|n| \,.
\end{aligned}
\tag{3.47}
$$

From the above expressions we can see that both components of the wave-function have different phase shifts, and these phase shifts present two distinct contributions: from the geometry of the spacetime itself codified by the parameter α, and by the anomalous factor κ. Due to its non-abelian character, the Aharonov-Bohm phase associated with the latter presents opposite sign. Because of the dynamics of each component does not depend on the other, their respective scattering amplitude can be calculated independently. Let us now develop these calculations using the partial-wave analysis:

$$
\delta_n^+ = \begin{cases} -n\dfrac{\omega}{2} - \dfrac{\upsilon^+}{2}, & n \geq -N\,, \\ n\dfrac{\omega}{2} + \dfrac{\upsilon^+}{2}, & n < -\,N \end{cases}
\tag{3.48}
$$

and

$$
\delta_n^- = \begin{cases} -n\dfrac{\omega}{2} + \dfrac{\upsilon^-}{2}, & n \geq \,N\,, \\ n\dfrac{\omega}{2} - \dfrac{\upsilon^-}{2}, & n < \,N\,, \end{cases}
\tag{3.49}
$$

where $\omega = \pi(\alpha^{-1} - 1)$ and $\upsilon^{\pm} = N\pi/\alpha \pm \omega/2 + \pi\delta/\alpha$. Here we shall adopt the traditional approach to procedure the summation. We separate the complete sum in n from $-N$ to ∞ and from $-\infty$ to $-N-1$ to the upper component, and from N to ∞ and from $-\infty$ to $N-1$ to the lower one, using the respective value to the phase shift in the specific intervals; moreover in evaluating the sums we replace θ by $\theta + i\epsilon$ in the first summations and by $\theta - i\epsilon$ in the second. All the summations can be done by usual way and after taken the limit $\epsilon \to 0$. Considering only scattering particle with $\theta \neq 0\,,\omega$ and $-\omega$, we get:

$$
f_R^+(\theta) = \frac{e^{-i(N+1/2)\theta}}{2i\sqrt{2\pi i k}} \left[\frac{e^{i\pi\delta/\alpha - iN\pi}}{\sin(\frac{\theta+\omega}{2})} - \frac{e^{-i\pi\delta/\alpha + iN\pi}}{\sin(\frac{\theta-\omega}{2})} \right]
\tag{3.50}
$$

and

$$
f_R^-(\theta) = \frac{e^{i(N-1/2)\theta}}{2i\sqrt{2\pi i k}} \left[\frac{e^{i\pi\delta/\alpha + iN\pi}}{\sin(\frac{\theta-\omega}{2})} - \frac{e^{-i\pi\delta/\alpha - iN\pi}}{\sin(\frac{\theta+\omega}{2})} \right].
\tag{3.51}
$$

Considering the case with $\alpha = 1$, $\omega = 0$, we have

$$
f_R^+(\theta) = (-1)^N \frac{e^{-i(N+1/2)\theta}}{\sqrt{2\pi i k}} \frac{\sin(\pi\delta)}{\sin(\theta/2)}
\tag{3.52}
$$

and

$$
f_R^-(\theta) = (-1)^N \frac{e^{i(N-1/2)\theta}}{\sqrt{2\pi i k}} \frac{\sin(\pi\delta)}{\sin(\theta/2)}\,.
\tag{3.53}
$$

JHEP06(2004)016

The expression (3.46) to the scattering amplitude, has been obtained by standard procedure which consists to substitute the asymptotic behavior for the Bessel function in (3.44); however it has been pointed out by Hagen [21] that for large values of the order of the Bessel function, this procedure becomes inappropriate. One has to sum up first and then take the limit $kr \to \infty$. Using the integral representation to the Bessel function [17]

$$J_\nu(z) = \frac{1}{\pi}\int_0^\pi dt\cos(\nu t - z\sin t) - \frac{\sin\nu t}{\pi}\int_0^\infty dt e^{-\nu t - z\sinh t}, \tag{3.54}$$

this approach can be applied. Because we are interested to calculate the scattering wave-function which behaves for large r as $e^{ikr}/\sqrt{r}$, we need only to use the integral $1/(2\pi) \times \int_0^\pi dte^{-i\nu t + iz\sin t}$. Substituting this integral in (3.44) and summing the geometric series according to the procedure indicate above, we obtain two integrals contributions for each component f_R^+ and f_R^-. Changing the variable t to $t' + \pi/2$ and taking the limit $kr \to \infty$, both integrals can be approximately evaluated providing for the scattering amplitude the same result as obtained before.

The contribution to the scattering amplitude due to (3.45) can also be calculated separately for both components. Their results are easily obtained:

$$f_I^+(\theta) = -(-1)^N e^{-i(N+1)\theta}\sqrt{\frac{2i}{\pi k}}\sin(\pi\beta) \tag{3.55}$$

and

$$f_I^-(\theta) = -(-1)^N e^{i(N+1)\theta}\sqrt{\frac{2i}{\pi k}}\sin(\pi\beta)\,. \tag{3.56}$$

4. Concluding remarks

In this paper we have explicitly analysed the non-relativistic quantum motion of a neutral particle possessing an anomalous magnetic moment in the presence of magnetic and electric fields in a conical spacetime. Two different physical systems have been investigated. The particle interacting with an azimuthal magnetic field and a radial electric one. In the first case we have shown that bound states between the neutral particle and uniform electric current may be formed, and the respective self-energy calculated. From our result, eq. (3.26), we observe that because $\alpha \neq 1$, the effective quantum number $2n + 1 + 2|j|/\alpha$ is not an integer one. This fact reduces the degree of degeneracy of this "Coulomb" problem. In the second analysis we have shown that the electric interaction, together with the geometric one, contribute only to the scattering phase shift. Also from (3.47) we can see that the scattering phase shifts associated with both components of the fermionic wave-function depend on the geometry itself by the parameter α and by the interaction between the anomalous magnetic moment and the electric field. Because of the non-abelian character associated with the latter, the scattering amplitude for both components present opposite sign in their respective phases for regular functions. Another point that we should mention is that Hagen has considered in his analysis about the Aharonov-Casher scattering [19] two

JHEP06(2004)016

degrees of freedom associated with the spin. Here we have explicitly use one. Including both degrees we would have reached the same conclusions as we did in sub-sections 3.1 and 3.2.

Bound states solutions for a neutral particle in a magnetic field of linear current density have been first investigated by Pron'ko and Stroganov [22] in a flat spacetime. There, exploring a hidden $O(3)$ dynamical symmetry to the system, the authors found an exact energy spectrum, which in our notation reads:

$$E_n = -\frac{2m\mu_B^2\kappa^2 j_0^2}{\hbar^2 c^2 n^2}\,, \tag{4.1}$$

being $n = 1, 2, 3 \ldots$. A few years latter Blümel and Dietrich [23] returned to this system, and considering the limit $|j| \gg 1$ in the set of differential equations, they found an approximated solution to the energy spectrum similar to ours, without the factor α. Moreover, using variational method they also showed that the improved results to the energy spectrum has no significant difference in the limit of large angular quantum number. Unfortunately, because we are considering this system in a conical spacetime, we could not find to the effective two-dimensional hamiltonian operator (3.8) the same degree of symmetry found in [22]. So only approximated results to the energy spectrum has been provided.

The main result of this analysis was to provide energy spectrum to a neutral particle in a magnetic field of a linear current density superposed to a conical singularity. This analysis may be useful to study the quantum motion of neutral spin$-1/2$ particle in disclinated medium in presence of electric current.

The analysis of Landau levels of charged particles in the presence of disclinations has been developed a few years ago [24]. There bound states were found, even considering the effect of a repulsive self-interaction. Here we obtained bound states for neutral particle as a consequence of the interaction between its anomalous magnetic moment with an azimuthal magnetic field.

Finally we would like to finish this paper by saying that a similar analysis, could also be develop if instead of an anomalous magnetic moment the particle had an anomalous electric dipole moment. In this case we had to to change $F_{(a)(b)}$ in (2.1) by its dual ${}^*F_{(a)(b)} = \frac{1}{2}\epsilon_{(a)(b)(c)(d)}F^{(c)(d)}$.

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A. Development of the non-relativistic hamiltonian

Here we shall present some intermediate steps used to obtain (2.18) from (2.17). First we need to find

$$\Pi_i \sigma^j = -i\hbar\nabla_i\sigma^j + \frac{e}{c}A_i\sigma^j\,, \tag{A.1}$$

with $\nabla_i = \partial_i + \Gamma_i$; however

$$\begin{aligned}\nabla_i \sigma^j &= \partial_i \sigma^j + \sigma^j \partial_i + \Gamma_i \sigma^j \\ &= \sigma^j \nabla_i + \partial_i \sigma^j + [\Gamma_i,\ \sigma^j]\,.\end{aligned} \tag{A.2}$$

So we need $\sigma^i(\partial_i \sigma^j)$. Writing $\sigma^i = e^i_{(a)}\sigma^{(a)}$, being $\sigma^{(a)}$ the constant Pauli matrix and $e^i_{(a)}$ the basis tetrad. We find

$$\sigma^i(\partial_i \sigma^j) = \frac{1}{\alpha r} g^{ij} + i(1-\alpha) g^{2j}\sigma^z\,. \tag{A.3}$$

We also need to find the commutator $[\Gamma_i,\ \sigma^j]$, where for this system the only non-vanishing spin connection is:

$$\Gamma_\phi = i(1-\alpha)\frac{\sigma^z}{2}\,. \tag{A.4}$$

So, $\sigma^i[\Gamma_i,\ \sigma^j] = i(1-\alpha)/2\sigma^\phi[\sigma^z,\ \sigma^j]$. Again, expressing the $\sigma-$matrices in the conical spacetime in terms of the constants one, and developing the the commutator, we get:

$$\sigma^i[\Gamma_i,\ \sigma^j] = -\frac{1-\alpha}{\alpha r} g^{1j} - i(1-\alpha) g^{2j}\sigma^z\,. \tag{A.5}$$

Adding (A.3) and (A.5) we obtain

$$\sigma^i(\partial_i \sigma^j) + \sigma^i[\Gamma_i,\ \sigma^j] = \frac{1}{r} g^{1j}\,. \tag{A.6}$$

After we have done this procedure, we have written (2.17) in terms of $\sigma^i\sigma^j(\Pi_i - i\kappa\mu_B/cE_i)(\Pi_j + i\kappa\mu_B/cE_j)$. Using the relation

$$\sigma^i\sigma^j = g^{ij} + i\frac{\epsilon^{ijm}}{\sqrt{g^{(3)}}} g_{mk}\sigma^k\,, \tag{A.7}$$

being $g^{(3)}$ the determinant of the space section of (1.1) and ϵ^{ijm} the totally anti-symmetric symbol with $\epsilon^{123} = 1$. We could have stopped our calculation at this point; however, in order to compare our expression with similar one obtained by Anandan in [12] to ordinary flat spacetime, we decided to present our result in terms of the modified linear operator $\Pi_i - \kappa(\mu_B/c)(\vec{E}\times\vec{\sigma})_i$. So we can observe that (2.18) is a generalization of the [12, eq. (8)] written in a conical spacetime.

References

[1] A. Vilenkin, *Cosmic strings and domain walls*, *Phys. Rept.* **121** (1985) 263.

[2] H. Kleinert, *Gauge field in condensed matter*, World Scientific, Singapore 1989.

[3] M. Kleman, *Points, lines and Walls*, Wiley, New York 1983.

[4] M.O. Katanaev and I.V. Volovich, *Theory of defects in solids and three-dimensional gravity*, *Ann. Phys. (NY)* **216** (1992) 1.

[5] W.A. Hiscock, *Exact gravitational field of a string*, *Phys. Rev.* **D 31** (1985) 3288;
B. Linet, *The static metric with cylindrical symmetry describing a model of cosmic string*, *Gen. Rel. Grav.* **17** (1985) 1109;
A. Vilenkin, *Gravitational field of vacuum domain walls and string*, *Phys. Rev.* **D 23** (1981) 852.

[6] B. Linet, *Force on a charge in the space-time of a cosmic string*, *Phys. Rev.* **D 33** (1986) 1833.

[7] A.G. Smith, *Gravitational effects of an infinite straight cosmic string on classical and quantum fields: self-forces and vacuum fluctuations*, in *Proceedings of symposium on the formation and evolution of cosmic strings*, G.W. Gibbons, S.W. Hawking and T. Vachaspati eds., Cambridge University Press, Cambridge, England, 1990.

[8] E.R. Bezerra de Mello, V.B. Bezerra, C. Furtado and F. Moraes, *Selfforces on electric and magnetic linear sources in the space-time of a cosmic string*, *Phys. Rev.* **D 51** (1995) 7140 [gr-qc/9503028].

[9] J.S. Dowker, *Gravitational Aharonov-Bohm effect*, *Nuovo Cim.* **52** (1967) 129;
L.H. Ford and A. Vilenkin, *A gravitational analogue of the Aharonov-Bohm effect*, *J. Phys. A* **14** (1981) 2353;
V.B. Bezerra, *Gravitational analog of the Aharonov-Bohm effect in four-dimensions and three-dimensions*, *Phys. Rev.* **D 35** (1987) 2031.

[10] V.P. Frolov and E.M. Serebryani, *Vacuum polarization in the gravitational field of a cosmic string*, *Phys. Rev.* **D 35** (1987) 3779;
D.D. Harari and V.D. Skarzhinsky, *Pair production in the gravitational field of a cosmic string*, *Phys. Lett.* **B 240** (1990) 322.

[11] D.D. Sokolov and A.A. Starobinsk, *Structure of the curvature tensor on conical singularities*, *Sov. Phys. Dokl.* **22** (1977) 312.

[12] J. Anandan, *Electromagnetic effects in the quantum interference of dipole*, *Phys. Lett.* **A 138** (1989) 347

[13] J. Audretsch and V.D. Skarzhinski, *Aharonov-Bohm scattering of a charged particle and neutral atoms: the role of absorption*, *Phys. Rev.* **A 60** (1999) 1854.

[14] M. Bordag and S. Voropaev, *Charged particle with magnetic moment in the Aharonov-Bohm potential*, *J. Phys. A* **26** (1993) 7637, [hep-th/9304017].

[15] M. Bordag and S. Voropaev, *Bound states and scattering of an electron in the field of the magnetic string*, *Phys. Lett.* **B 333** (1994) 238.

[16] J.D. Bjorken and S.D. Drell, *Relativistic Quantum Mechanics*, McGraw-Hill, New York, 1964, p. 241

[17] I.S. Gradshteyn and I.M. Ryzhik, *Tables of integrals, series, and products*, Academic Press, New York, 1980;
Handbook of mathematical functions, 9th ed., M. Abramowitz and I.A. Stegum eds., Dove, New York, 1972

[18] C.R. Hagen, *Aharonov-Bohm scattering of particles with spin*, *Phys. Rev. Lett.* **64** (1990) 503.

[19] C.R. Hagen, *Exact equivalence of spin 1/2 Aharonov-Bohm and Aharonov-Casher effects*, *Phys. Rev. Lett.* **64** (1990) 2347.

[20] S.A. Voropaev and M. Bordag, *The role of boundary conditions in the Aharonov-Bohm effect for particles with spin*, *J. Exp. Theor. Phys* **78** (1994) 127.

[21] C.R. Hagen, *The Aharonov-Bohm scattering amplitude*, *Phys. Rev.* **D 41** (1990) 2015.

[22] G.P. Pron'ko and Yu. G. Stroganov, *A new example of quantum mechanical problem with a hidden symmetry*, *Sov. Phys. JETP* **45** (1977) 1075.

[23] R. Blümel and K. Dietrich, *Bound states of neutros in the cylindrically symmetric magnetic field of a thin current carrying wire*, *Phys. Lett.* **A 139** (1989) 236.

[24] C. Furtado, B.G.C. da Cunha, F. Moraes, E.R. Bezerra de Mello and V.B. Bezerra, *Landau levels in the presence of disclinations*, *Phys. Lett.* **A 195** (1994) 90.